



# On central extensions of preprojective algebras

Pavel Etingof<sup>a,\*</sup>, Frédéric Latour<sup>b</sup>, Eric Rains<sup>c</sup>

<sup>a</sup> *Department of Mathematics, Room 2-176, Massachusetts Institute of Technology, 77 Massachusetts Ave., Cambridge, MA 02139, USA*

<sup>b</sup> *Department of Mathematics, Central Connecticut State University, 1615 Stanley Street, New Britain, CT 06050, USA*

<sup>c</sup> *Mathematical Sciences Building, One Shields Ave., University of California, Davis, CA 95616, USA*

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## Abstract

We show that the Hilbert polynomial  $P(t)$  of the trace space  $A/[A, A]$  of the centrally extended preprojective algebra  $A$  of an ADE quiver is equal to the Hilbert series of the maximal nilpotent subalgebra of the corresponding simple Lie algebra under the principal gradation. This implies that the Hilbert polynomial of the center of  $A$  is  $t^{2h-4}P(1/t)$ , where  $h$  is the Coxeter number.

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## 1. Introduction

Let  $Q$  be a quiver of ADE type. Let  $\bar{Q}$  be the double of  $Q$ , and  $P$  the path algebra of  $\bar{Q}$  over  $\mathbb{C}$ . The paper [ER] attaches to  $Q$  a centrally extended preprojective algebra  $A = A^\mu$ , which is the quotient of  $P[z]$  by the relation  $\sum_{a \in Q} [a, a^*] = z(\sum \mu_i e_i)$ , where  $\mu = (\mu_i)$  is a regular weight (for the root system attached to  $Q$ ), and  $e_i$  are the vertex idempotents in  $P$ .<sup>1</sup> It is shown in [ER] that the algebra  $A$  has nicer properties than the ordinary Gelfand–Ponomarev preprojective

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\* Corresponding author.

*E-mail addresses:* etingof@math.mit.edu (P. Etingof), flatour@mailbox.syr.edu (F. Latour), rains@math.ucdavis.edu (E. Rains).

<sup>1</sup> This algebra is denoted in [ER] by  $\Pi_0^\mu$ .

algebra  $A_0 = A/(z)$  of  $Q$ ; in particular, the deformed version  $A(\lambda)$  of  $A = A(0)$  is flat, while this is not the case for  $A_0$ . The paper [ER] also shows that  $A$  is a Frobenius algebra, and computes the Hilbert series of  $A$ . Finally, [ER] links the algebra  $A$  with cyclotomic Hecke algebras of complex reflection groups of rank 2.

The goal of this paper is to continue to study the rich structure of the algebra  $A$ . In particular, we show that for generic  $\mu$  (and specifically for  $\mu = \rho$ ) the algebra  $A$  has a unique trace functional, and compute the structure of the center  $Z$  of  $A$  and the trace space  $A/[A, A]$ . Namely, it turns out that  $Z$  and  $A/[A, A]$  are dual to each other under the trace form, and the dimension of the homogeneous subspace  $(A/[A, A])[2p]$  equals the number of positive roots for  $Q$  of height  $p + 1$ .

We also show that the elements  $z^s(\sum \phi_i e_i)$  span  $A/[A, A]$ , and determine when such an element maps to zero in  $A/[A, A]$  (i.e. sits in  $[A, A]$ ). The answer is given in terms of the structure of the maximal nilpotent subalgebra  $\mathfrak{n}$  of the simple Lie algebra  $\mathfrak{g}$  attached to  $Q$ , which demystifies the equality between  $\dim(A/[A, A])[2p]$  and the number of positive roots for  $Q$  of height  $p + 1$ . This provides an interesting new connection between quivers and Lie algebras, which we do not fully understand at this time.

We note that the results of this paper were generalized in the paper [Eu], in which higher Hochschild homology and cohomology of the centrally extended preprojective algebras is computed. It has 4-periodicity, and expresses via the spaces  $Z$  and  $A/[A, A]$  computed in this paper.

## 2. Preliminaries and some results

### 2.1. Preliminaries

We recall some definitions and notation from [ER].

Let  $Q$  be a quiver of ADE type. Let  $I$  be the set of vertices of  $Q$ , and  $r = |I|$ .

Consider the root system  $\mathcal{R}$  attached to  $Q$ . Let  $\alpha_j$ ,  $j \in I$ , be the simple roots. Let  $\omega_j$ ,  $j \in I$ , be the fundamental weights. Let  $\rho = \sum \omega_i$ . If  $\alpha$  is a positive root, then the height of  $\alpha$  is the number of simple roots occurring in the decomposition of  $\alpha$ ; it equals to the inner product  $(\rho, \alpha)$ . Let  $h$  be the Coxeter number of  $\mathcal{R}$ . Let  $N$  be the number of positive roots in  $\mathcal{R}$ . Recall that  $N = hr/2$ .

Let  $\mathfrak{g}$  be the simple Lie algebra whose Dynkin diagram is  $Q$ . Fix a polarization  $\mathfrak{g} = \mathfrak{n}_+ \oplus \mathfrak{h} \oplus \mathfrak{n}_-$ , where  $\mathfrak{n}_\pm$  are the nilpotent subalgebras, and  $\mathfrak{h}$  the Cartan subalgebra. For brevity we will denote  $\mathfrak{n}_-$  by  $\mathfrak{n}$ . The Lie algebra  $\mathfrak{n}$  is generated by elements  $F_i$ ,  $i \in I$ , subject to the Serre relations.

Let  $R$  be the algebra of complex-valued functions on  $I$ , and  $e_i$ ,  $i \in I$ , be the primitive idempotents of this algebra. Let  $\overline{Q}$  be the double of  $Q$ . Let  $V$  be the  $R$ -bimodule spanned by the edges of  $\overline{Q}$ . Let  $P = T_R V$  be the path algebra of the doubled quiver  $\overline{Q}$  (the tensor algebra over  $R$  of the bimodule  $V$ ). Let  $\mu = \sum_{i \in I} \mu_i \omega_i \in \mathfrak{h}^*$  be a regular weight (i.e. the inner product  $(\mu, \alpha) \neq 0$  for any root  $\alpha \in \mathcal{R}$ ). Define the centrally extended preprojective algebra  $A = A^\mu$  of  $Q$ , which is the quotient of  $P[z]$  (where  $z$  is a central variable) by the relation

$$\sum_{a \in Q} [a, a^*] = z \left( \sum_{i \in I} \mu_i e_i \right).$$

Note that if  $\mu = \rho$  then this relation takes an especially simple form

$$\sum_{a \in Q} [a, a^*] = z.$$

Also, let  $A_0 := A/(z)$  be the usual preprojective algebra of  $Q$  (it is the quotient of  $P$  by the relation  $\sum_{a \in Q} [a, a^*] = 0$ ).

Define the deformed centrally extended preprojective algebra  $A(\lambda) = A^\mu(\lambda)$  to be the quotient of the path algebra  $P[z]$  by the defining relation

$$\sum_{a \in Q} [a, a^*] = \sum_{i \in I} (\mu_i z + \lambda_i) e_i,$$

where  $\lambda = \sum_{i \in I} \lambda_i \omega_i \in \mathfrak{h}^*$  is a weight. This algebra carries a natural filtration, given by  $\deg(R) = 0$ ,  $\deg(a) = \deg(a^*) = 1$ ,  $\deg(z) = 2$ . It is shown in [ER] that  $A(\lambda)$  is a flat deformation of  $A(0) = A$ , i.e.,  $\text{gr}(A(\lambda)) = A(0)$ .

It is clear that the algebras  $A_0$  and  $A(\lambda)$  are independent on the orientation of  $Q$ , up to an isomorphism.

## 2.2. The trace function on $A$

From now on we assume that  $\mu$  is a fixed generic weight, or  $\mu = \rho$ .

Recall from [ER] that  $A$  is a finite-dimensional  $\mathbb{Z}_+$ -graded Frobenius algebra, with socle in degree  $2(h-2)$ , with basis  $z^{h-2}e_i$ .<sup>2</sup>

### Proposition 2.1.

- (i) *There exists a unique up to scaling trace  $\text{Tr}: A \rightarrow \mathbb{C}$  of degree  $2(h-2)$ , i.e. a nonzero linear functional such that  $\text{Tr}(xy) = \text{Tr}(yx)$ .*
- (ii) *The form  $(x, y) := \text{Tr}(xy)$  is nondegenerate.*

**Proof.** Clearly, we may assume that  $Q$  has at least two vertices. Recall that the degree 1 component  $A[1]$  of  $A$  is spanned by edges  $a$  of the doubled quiver  $\overline{Q}$ . Also,  $A[2(h-2)-1]$  is spanned by elements of the form  $z^{h-3}b$ , where  $b$  is an edge of  $\overline{Q}$ . Indeed, it follows from [ER, Section 4], that this is true for  $\mu = \rho$ , hence it is true for generic  $\mu$  by a deformation argument.

Since  $A$  is a Frobenius algebra, for every edge  $a$  we have  $z^{h-3}aa^* = c_a z^{h-2}e_{\text{head}(a)}$ , where  $c_a$  is a nonzero number.

But  $[A, A][2(h-2)] = [A[1], A[2(h-2)-1]]$ , so it is the span of  $z^{h-3}[a, a^*]$  for the edges  $a \in Q$ , i.e. of elements  $z^{h-2}(c_a e_{\text{head}(a)} - c_{a^*} e_{\text{tail}(a)})$ . It is clear that these elements span a subspace of codimension 1 in  $A[2(h-2)]$ ; thus the functional  $\text{Tr}$  is unique up to scaling; it is a linear functional of top degree that vanishes on this codimension 1 subspace. Moreover,  $\text{Tr}(z^{h-2}e_i)$  is clearly nonzero for any  $i$ . The proposition is proved.  $\square$

Now let  $Z$  be the center of  $A$ .

**Corollary 2.2.** *The inner product  $(x, y)$  defines a nondegenerate pairing  $Z \times A/[A, A] \rightarrow \mathbb{C}$ .*

**Proof.** The statement is well known but we give a proof for completeness. If  $x \in Z$  and  $y = [y_1, y_2] \in [A, A]$  then  $(x, y) = \text{Tr}(x[y_1, y_2]) = \text{Tr}([xy_1, y_2]) = 0$ . Thus the pairing in question is well defined. To show that it is nondegenerate, by Proposition 2.1(ii), it suffices to show that  $Z^\perp \subset [A, A]$ , or equivalently,  $Z \supset [A, A]^\perp$ .

<sup>2</sup> Note that the elements  $z^{h-2}e_i$  may vanish for special regular  $\mu$ .

The latter statement is obvious. Indeed, if  $\text{Tr}(x[y_1, y_2]) = 0$  for any  $y_1, y_2$ , then  $\text{Tr}([x, y_1]y_2) = 0$  for any  $y_1, y_2$ , and therefore  $[x, y_1] = 0$  for all  $y_1$ , implying  $x \in Z$ .  $\square$

Let  $p(t) = \sum \dim(A/[A, A])[m]t^m$  be the Hilbert polynomial of  $A/[A, A]$ , and  $p_*(t) = \sum \dim Z[m]t^m$  be the Hilbert polynomial of  $Z$ .

**Corollary 2.3.** *The polynomials  $p, p_*$  are reverses of each other, i.e.  $p_*(t) = t^{2(h-2)}p(1/t)$ .*

### 2.3. The spaces $Z$ and $A/[A, A]$ as $\mathbb{C}[z]$ -modules

Let  $E$  be the subspace of  $A$  spanned by elements  $z^j e_i$ . Obviously, it has dimension  $(h-1)r$ . The Hilbert polynomial of  $E$  is  $\frac{1-t^{2h-2}}{1-t^2}r$ .

**Proposition 2.4.** *The natural map  $\psi: E \rightarrow A/[A, A]$  is surjective.*

**Proof.** It is shown in [MOV, Section 4], that  $A_0/[A_0, A_0]$  is freely spanned by the idempotents  $e_i$ . This implies that if  $x \in A$  is an element of positive degree  $d$  then there exists a homogeneous element  $y \in A$  of degree  $d-2$  such that  $x - zy \in [A, A]$ . Thus the statement follows by induction in  $d$ .  $\square$

Note now that  $A/[A, A]$  and  $Z$  are naturally  $\mathbb{C}[z]$ -modules, and the pairing  $(,)$  between them is invariant in the sense that the operator of multiplication by  $z$  is selfadjoint.

**Corollary 2.5.** *The  $\mathbb{C}[z]$ -module  $A/[A, A]$  is minimally generated by  $e_i$ .*

**Proof.** Indeed,  $A/[A, A]$  is a quotient of  $E$  by the submodule  $E \cap [A, A]$ , which shows that it is generated by  $e_i$ . The minimality of this set of generators is obvious.  $\square$

Thus we see that the operator  $z$  in  $A/[A, A]$  and  $Z$  is a direct sum of  $r$  nilpotent Jordan blocks, of some sizes  $m_1 \leq m_2 \leq \dots \leq m_r$ , and  $p(t) = \sum_{i=1}^r \frac{1-t^{2m_i}}{1-t^2}$ .

## 3. The main theorem

Let  $N_p$  be the number of positive roots for  $Q$  of height  $p+1$ .  
One of our main results is the following theorem.

### Theorem 3.1.

- (i)  $\dim Z = \dim(A/[A, A]) = N$ .
- (ii) *The sizes  $m_i$  of the Jordan blocks of  $z$  on  $Z$  and  $A/[A, A]$  are the exponents of the root system attached to  $Q$ . In other words, we have  $\dim(A/[A, A])[2p] = N_p$  for all  $p \geq 0$ .*

The proof of Theorem 3.1 is given in the next two subsections.

#### 3.1. The lower bound

**Proposition 3.2.**  $\dim Z \geq N$ .

**Proof.** According to [ER], for generic  $\lambda$  the algebra  $A(\lambda)$  is semisimple with irreducible representations  $V_\alpha$  corresponding to positive roots  $\alpha$ . This implies that the center  $Z(\lambda)$  of  $A(\lambda)$  is a semisimple algebra of dimension  $N$ . Since  $\text{gr}(A(\lambda)) = A$ , we have  $\text{gr}(Z(\lambda)) \subset Z$ , and we get the desired inequality.  $\square$

### 3.2. The upper bound

We have  $\sum_p N_p = N$ . Therefore, by Proposition 3.2, to prove Theorem 3.1, it suffices to show that  $\dim(A/[A, A])[2p] \leq N_p$  for all  $p \geq 0$ .

We do it case by case, following the idea of the argument of [MOV, Section 4]. Since we need to establish the result for generic  $\mu$ , it suffices to consider the case  $\mu = \rho$ .

**Case 1.** Type  $A_n$ . In this case  $N_p = \max(n - p, 0)$ . Denote the corresponding algebra  $A$  by  $A^n$ , and let us prove the desired statement by induction in  $n$ .

The base of induction ( $n = 1$ ) is obvious, so let us perform the induction step. Assume that the statement is known for  $A^{n-1}$ , and let us prove it for  $A^n$  ( $n \geq 2$ ).

Let  $J = Ae_nA$  be the ideal in  $A^n$  spanned by paths passing through the end-vertex  $n$  of the Dynkin diagram. Then  $A^n/J = A^{n-1}$ . Thus, using the induction assumption, we see that

$$\dim A^n/(J + [A^n, A^n])[2p] \leq \max(n - 1 - p, 0).$$

Thus to establish the induction step (i.e. to show that  $\dim(A^n/[A^n, A^n]) \leq n - p$ ), it suffices to show that  $\dim(J/(J \cap [A^n, A^n]))[2p] \leq 1$  for  $p \leq n - 1$  and is zero if  $p \geq n$ .

Define the algebra  $B_n := e_n A^n e_n$ . It is easy to see that the natural map  $\phi: B_n \rightarrow J/(J \cap [A^n, A^n])$  is surjective. On the other hand, it is easy to check that  $B_n$  is a commutative  $n$ -dimensional algebra:  $B_n = \mathbb{C}[z]/(z^n)$ . Thus the desired statement follows.

**Case 2.** Types  $D$ ,  $E$ . Let  $*$  be the nodal vertex of the Dynkin diagram, and  $J = Ae_*A$ . Then  $A/J = A^{\ell_1} \oplus A^{\ell_2} \oplus A^{\ell_3}$ , where  $\ell_1, \ell_2, \ell_3$  are the lengths of the three legs of the Dynkin diagram. Thus by Case 1,

$$\dim A/(J + [A, A])[2p] \leq \sum_{j=1}^3 \max(\ell_j - p, 0).$$

So it suffices to show that

$$\dim J/(J \cap [A, A])[2p] \leq N'_p,$$

where  $N'_p := N_p - \sum_{j=1}^3 \max(\ell_j - p, 0)$  is the number of positive roots of height  $p + 1$  which contain the simple root  $\alpha_*$  in their expansion.

Define the algebra  $B := e_* A e_*$ . It is easy to see that the natural map  $\phi: B \rightarrow J/(J \cap [A, A])$  is surjective. Thus, it suffices to show that

$$\dim(B/[B, B])[2p] \leq N'_p.$$

According to [ER], the algebra  $B$  is generated by degree 2 elements  $U_1, U_2, U_3$  with defining relations

$$U_1 + U_2 + U_3 = z, \quad [z, U_i] = 0, \quad \prod_{m=0}^{\ell_i} (U_i + mz) = 0, \quad i = 1, 2, 3. \quad (3.1)$$

**Case 2a.** Type  $D_{n+2}$  ( $n \geq 2$ ). We have  $\ell_1 = \ell_2 = 1, \ell_3 = n - 1$ . So, setting  $a = U_1 + z/2, b = U_2 + z/2$ , we have the following defining relations for  $B$ :

$$a^2 = b^2 = z^2/4, \quad [a, z] = [b, z] = 0, \\ (a + b - 2z)(a + b - 3z) \dots (a + b - (n+1)z) = 0.$$

Let  $a_s = aba \dots, b_s = bab \dots$  (words of length  $s$ ).

**Lemma 3.3.** *If  $p < n$  then a basis of  $B[2p]$  is formed by the words  $a_s z^{p-s}, b_s z^{p-s}, p \geq s > 0$ , and  $z^p$ . If  $p = n$ , then a basis of  $B[2p]$  is formed by the words  $a_s z^{p-s}, b_s z^{p-s}, n > s > 0, a_n$  and  $z^n$ . If  $p > n$ , then a basis of  $B[2p]$  is formed by the words  $a_s z^{p-s}, b_s z^{p-s}, 2n - p \geq s > 0$ , and  $z^p$ .*

**Proof.** It is easy to see from the relations that these words are a spanning set for  $B[2p]$ . The fact that they are linearly independent follows from the Hilbert series formula for  $B$  given in [ER].  $\square$

**Lemma 3.4.** *One has  $\dim(B/[B, B])[2p] \leq N'_p$ .*

**Proof.** It is straightforward to show (by explicit inspection of the root system of type  $D_{n+2}$ ) that  $N'_0 = 1, N'_p = 3 + [p/2]$  if  $1 \leq p \leq n - 1, N'_n = 2 + [n/2]$ , and  $N'_p = 1 + [n - p/2]$  for  $p > n$ , where  $[x]$  is the integer part of  $x$ . For odd  $s > 1$  and  $p \geq s$ , we have

$$a_s z^{p-s} = \frac{1}{4} b_{s-2} z^{p-s+2} + [a, b_{s-1} z^{p-s}], \\ b_s z^{p-s} = \frac{1}{4} a_{s-2} z^{p-s+2} + [b, a_{s-1} z^{p-s}].$$

Also, for even  $s > 0$ ,

$$(a_s - b_s) z^{p-s} = [a, b_{s-1} z^{p-s}].$$

This together with Lemma 3.3 implies that  $(B/[B, B])[2p]$  is spanned by  $z^p, a z^{p-1}, b z^{p-1}$ , and  $a_s z^{p-s}$  for even  $s > 0$ . Hence, for  $p \geq 1$  we have

$$\dim(B/[B, B])[2p] \leq 3 + [p/2],$$

i.e. the lemma is proved for  $p < n$ . Moreover, for  $p = n$  the last relation of  $B$  implies that  $z^n$  is a linear combination of  $a_s z^{n-s}$  and  $b_s z^{n-s}$  for  $s > 0$ , which implies that

$$\dim(B/[B, B])[2n] \leq 2 + [n/2],$$

i.e. the lemma is also proved for  $p = n$ .

Now let us prove the lemma for  $p > n$ . Let  $Z_B$  be the center of  $B$ . The pairing  $(x, y)$  of Proposition 2.1 has degree  $4n$ . Therefore, by Proposition 2.1 (similarly to Corollary 2.2), it suffices to show that  $\dim Z_B[2k] \leq 1 + [k/2]$  for  $k < n$ . In showing this, we can obviously ignore the last relation of  $B$  (which has degree  $2n$ ). In other words, we should consider the algebra  $B'$  with generators  $a, b, z$  and relations  $a^2 = b^2 = z^2/4$ ,  $[z, a] = [z, b] = 0$ . It is easy to see that a basis in  $B'$  is formed by elements  $z^p(a+b)^{2q}$ ,  $az^p(a+b)^{2q}$ ,  $bz^p(a+b)^{2q}$ ,  $abz^p(a+b)^{2q}$ ,  $p, q \geq 0$ , and thus the center of  $B'$  is spanned by  $z^p(a+b)^{2q}$ , which implies the desired inequality.  $\square$

**Case 2b.** Types  $E_6, E_7, E_8$ . Using the presentation (3.1) of  $B$  and the Magma code by the third author [Mag], one determines, by a direct computer calculation, that  $\dim(B/[B, B])[2p] = N'_p$ .

Theorem 3.1 is proved.

### 3.3. Derivations of $A$

Theorem 3.1 implies the following result.

**Corollary 3.5.** *Every derivation of  $A$  which annihilates  $R$  and  $z$  is inner.*

**Remark.** In this corollary, we can omit the hypothesis that the derivation annihilates  $R$ . Indeed, for any derivation  $D$  of  $A$ , if we let  $u_D := \sum_{i \in I} e_i D(e_i)$ , then  $D + \text{Ad}(u_D)$  annihilates  $R$  and has the same action as  $D$  on the central element  $z$ .

**Proof.** We consider the complex of graded vector spaces

$$0 \rightarrow D_0 \rightarrow D_1 \rightarrow D_2 \rightarrow 0,$$

with differentials  $d_i: D_i \rightarrow D_{i+1}$ , where  $D_0 = A^R[2]$ ,  $D_1 = (A \otimes_R V)^R$ ,  $D_2 = A^R$  (where the superscript  $R$  denotes the  $R$ -invariants in a bimodule, and  $[i]$  denotes the shift of grading), and

$$\begin{aligned} d_0(x) &= \sum_{a \in Q} ([x, a] \otimes a^* - [x, a^*] \otimes a), \\ d_1(y \otimes b) &= [y, b]. \end{aligned}$$

It is clear that these differentials have degree 0. The fact that  $d_1 \circ d_0 = 0$  follows from the fact that  $z$  is a central element.

Let  $H_0, H_1, H_2$  be the homology groups of the complex  $D_\bullet$ . Then we have  $H_0 = Z[2]$ ,  $H_2 = A/[A, A]$ .

Let  $q(t)$  be the Hilbert polynomial of  $H_1$ . Then, computing the Euler characteristic in each homogeneous component of  $D_\bullet$ , we obtain the following identity for Hilbert polynomials:

$$t^2 p_*(t) + p(t) - q(t) = \text{Tr}((1 - Ct + t^2)h(t)),$$

where  $h(t)$  is the (matrix-valued) Hilbert polynomial of  $A$ , and  $C$  is the adjacency matrix of  $\overline{Q}$ . But it is proved in [ER] that

$$h(t) = \frac{1 - t^{2h}}{1 - t^2} (1 - Ct + t^2)^{-1}.$$

This implies that

$$q(t) = t^2 p_*(t) + p(t) - \frac{1 - t^{2h}}{1 - t^2} r.$$

Now recall that the exponents of a root system satisfy the equality  $m_{r+1-i} = h - m_i$ . This implies that  $t^2 p_*(t) + p(t) = \frac{1-t^{2h}}{1-t^2} r$ , and hence  $q(t) = 0$ . Thus  $H_1 = 0$ .

Now let  $D$  be a derivation of  $A$  which annihilates  $R$  and  $z$ . Let  $x_D := \sum_{a \in Q} (Da \otimes a^* - Da^* \otimes a)$ . Then  $d_1 x_D = 0$ . Since  $H_1 = 0$ , this implies that  $x_D = d_0 y$ , i.e.  $D = \text{ad } y$ , as desired. The corollary is proved.  $\square$

#### 4. Relation to simple Lie algebras

The computer assisted case-by-case proof of Theorem 3.1 makes it look mysterious (especially part (ii)). The results of this section demystify this theorem, by making explicit the relation of the structure of  $A/[A, A]$  with that of the maximal nilpotent subalgebra of the simple Lie algebra corresponding to  $Q$ .

##### 4.1. The results

Let us color the vertices of  $Q$  white and black so that every edge connects a white vertex with a black vertex. Let  $\varepsilon_i$  be  $+1$  for white vertices  $i$  and  $-1$  for black vertices. Let  $F = \sum_i \varepsilon_i F_i$  be a principal nilpotent element.

Let  $h_\lambda \in \mathfrak{h}$  be the element corresponding to the weight  $\lambda \in \mathfrak{h}^*$  under the standard inner product on  $\mathfrak{h}^*$  normalized so that  $(\alpha, \alpha) = 2$  for roots  $\alpha$ .

The following theorem characterizes explicitly the space  $E \cap [A, A]$ .

**Theorem 4.1.** *Let  $\phi_i, i \in I$  be complex numbers, and  $s \geq 0$  be an integer. Then the element  $z^s (\sum_i \varepsilon_i \phi_i e_i)$  is in  $[A, A]$  if and only if*

$$(\text{ad}(F)\text{ad}(h_\mu)^{-1})^s \left( \sum \phi_i F_i \right) = 0$$

in  $\mathfrak{n}$ .

The result of Theorem 4.1 can be stated more explicitly as follows.

Let  $V_i$  be the space of complex-valued functions on the set of positive roots for the quiver  $Q$  of height  $i$  (i.e. sums of  $i$  simple roots). Define the operator  $T_i : V_i \rightarrow V_{i+1}$  by the formula

$$(T_i f)(\alpha) = \sum_{j: (\alpha_j, \alpha) = 1} \frac{f(\alpha - \alpha_j)}{(\mu, \alpha - \alpha_i)}.$$

(Note that  $\alpha - \alpha_j$  is a root iff  $(\alpha, \alpha_j) = 1$ .)

**Theorem 4.2.** *Let  $\phi \in V_1$ ,  $\phi_i = \phi(\alpha_i)$ . Let  $s \geq 0$ . Then the element  $z^s (\sum_i \varepsilon_i \phi_i e_i)$  is in  $[A, A]$  iff  $T_s T_{s-1} \dots T_1 \phi = 0$ .*



**Proof.** According to [Lu], there is a Chevalley basis  $\{F_\alpha\}$  of  $\mathfrak{n}$  normalized in such a way that  $[F_i, F_\alpha] = \varepsilon_i F_{\alpha+\alpha_i}$  provided  $\alpha + \alpha_i$  is a root. Therefore,

$$\mathrm{ad}(F)\mathrm{ad}(h_\mu)^{-1} \sum_{\beta \in \mathcal{R}: (\rho, \beta)=d} \phi_\beta F_\beta = \sum_{\gamma \in \mathcal{R}: (\rho, \gamma)=d+1} \sum_i \frac{\phi_{\gamma-\alpha_i}}{(\mu, \gamma - \alpha_i)} F_\gamma.$$

Thus Theorem 4.1 implies Theorem 4.2.  $\square$

**Corollary 4.3.** *The explicit form of the trace functional for  $A$  is*

$$\mathrm{Tr}(z^{h-2}e_i) = \varepsilon_i T_{h-2} \dots T_1 u_i,$$

where  $u_i \in V_1$  is such that  $u_i(\alpha_j) = \delta_{ij}$ .

This formula can be written more explicitly as follows. Let  $\theta$  be the maximal root of  $\mathcal{R}$ . Define a path in  $\mathcal{R}$  to be a sequence of positive roots  $\beta_1, \beta_2, \dots, \beta_m$  such that  $\beta_{i+1} - \beta_i = \alpha_{j_i}$  for some  $j_i$ . Define the weight of such a path  $\pi$  to be

$$w_\mu(\pi) = \prod_{i=1}^{m-1} (\mu, \beta_i)^{-1}.$$

Then we get

$$\mathrm{Tr}(z^{h-2}e_i) = \varepsilon_i \sum_{\pi} w_\mu(\pi),$$

where the summation is taken over all paths  $\pi$  which start at  $\alpha_i$  and end at  $\theta$  (so they have length  $h-1$ ). In particular, if  $\mu = \rho$  then after renormalization we get

$$\mathrm{Tr}(z^{h-2}e_i) = \varepsilon_i n_i,$$

where  $n_i$  is the number of paths leading from  $\alpha_i$  to  $\theta$ .

#### 4.2. Proof of Theorem 4.1

Let  $W(\lambda)$  be the space of collections of polynomials  $f_i, i \in I$  of degree  $\leq h-2$ , such that  $\sum_{i \in I} f_i(z) \varepsilon_i e_i \in [A(\lambda), A(\lambda)]$ .

**Proposition 4.4.** *Let  $\lambda$  be generic. Let  $f_i, i \in I$ , be polynomials of degree  $\leq h-2$ . Then  $\{f_i, i \in I\}$  belongs to  $W(\lambda)$  iff*

$$\sum f_i \left( -\frac{(\lambda, \alpha)}{(\mu, \alpha)} \right) \varepsilon_i(\alpha, \omega_i) = 0$$

for all positive roots  $\alpha$ .

**Proof.** Let us calculate the trace of  $\sum f_i(z)\varepsilon_i e_i$  in the irreducible representation  $V_\alpha$  of  $A(\lambda)$  whose dimension vector is  $\alpha$ . Since  $z$  acts on this representation by the scalar  $-(\lambda, \alpha)/(\mu, \alpha)$ , we get the statement.  $\square$

**Proposition 4.5.** *Let  $\lambda$  be generic. Let  $f_i$ ,  $i \in I$ , be polynomials of degree  $\leq h - 2$ . Then  $\{f_i, i \in I\}$  belongs to  $W(\lambda)$  iff*

$$\sum_{i \in I} f_i(\text{ad}(-h_\lambda + F)\text{ad}(h_\mu)^{-1})F_i = 0.$$

**Proof.** The linear operator  $L := \text{ad}(-h_\lambda + F)\text{ad}(h_\mu)^{-1}$  on  $\mathfrak{n}$  has eigenvalues  $-(\lambda, \alpha)/(\mu, \alpha)$ , where  $\alpha$  ranges over positive roots; let the corresponding eigenvectors be  $v_\alpha$ . Then, if we write

$$F_i = \sum_{\alpha} c_i(\alpha)v_\alpha,$$

we find

$$\sum_{i \in I} f_i(\text{ad}(-h_\lambda + F)\text{ad}(h_\mu)^{-1})F_i = \sum_{i \in I} \sum_{\alpha} f_i \left\{ \left( -\frac{(\lambda, \alpha)}{(\mu, \alpha)} \right) \right\} c_i(\alpha)v_\alpha.$$

In particular, to prove the proposition, it will suffice to show that

$$c_i(\alpha) \propto \varepsilon_i(\alpha, \omega_i)$$

(where  $\propto$  denotes proportionality).

Now, by duality,  $c_i(\alpha)$  can be computed as the coefficient of  $E_i \in \mathfrak{n}_+$  in the expansion of the eigenvector  $v_\alpha^*$  of the dual operator

$$L^* = \text{ad}(h_\mu)^{-1}\text{ad}(-h_\lambda + F)$$

on  $\mathfrak{g}$  with eigenvalue  $-(\lambda, \alpha)/(\mu, \alpha)$ .

Let  $X_\alpha \in \mathfrak{n}_+$  be the projection of  $v_\alpha^*$  to  $\mathfrak{n}_+$  along  $\mathfrak{h} \oplus \mathfrak{n}_-$ . Then the element

$$y := \left[ -h_\lambda + F + \frac{(\lambda, \alpha)}{(\mu, \alpha)} h_\mu, X_\alpha \right]$$

must belong to the Cartan subalgebra  $\mathfrak{h}$ .

Recall that  $\lambda \in \mathfrak{h}^*$  is generic. Therefore, if  $v \in \mathfrak{h}^*$  is any element of the orthogonal complement of  $\alpha$ , then there exists  $\mathcal{N} \in \mathfrak{n}$  such that

$$\left[ -h_\lambda + F + \frac{(\lambda, \alpha)}{(\mu, \alpha)} h_\mu, h_v + \mathcal{N} \right] = 0,$$

and thus  $(y, h_v + \mathcal{N}) = 0$ . It follows that  $y \propto h_\alpha$ ; since  $X_\alpha$  was only determined up to scale, we may as well insist that  $y = h_\alpha$ .

Since  $X_\alpha = \sum c_i(\alpha) E_i + \text{lower terms}$ , we find that

$$h_\alpha = [F, X_\alpha]_{\mathfrak{h}} = \sum \varepsilon_i c_i(\alpha) h_{\alpha_i}$$

(where the subscript  $\mathfrak{h}$  denotes the  $\mathfrak{h}$ -part), and thus  $\varepsilon_i c_i(\alpha) = (\alpha, \omega_i)$  as required.  $\square$

Now we can finish the proof of Theorem 4.1. For this, note that by Theorem 3.1(i), the space  $W(0)$  is the limit of spaces  $W(\lambda)$  as  $\lambda \rightarrow 0$ . Therefore, Proposition 4.5 implies Theorem 4.1.

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